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On the classical Yang–Baxter equation for Clifford sequences

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Abstract. We first consider a vector space V, the endomorphism ring $End(V)$ and a multiplicative pre-Poisson structure on $End(V)$. We describe a procedure of induction of multiplicative pre-Poisson structures on $End(End(V))$, $End(End(End(V)))$, ... using Grassmann variables.

Secondly, we show that these induced structures satisfy the Jacobi identity if and only if $End(V)$ carries a triangular Poisson structure. Therefore we obtain an obstruction for the Jacobi identity on $GL(n^2)$ for the standard multiplicative Poisson structure on $GL(n)$.

Non-commutative geometry can be viewed as the quantum deformation of an increasing set of mathematical objects and theories. In the quasi-classical limit we have the corresponding problem of the classification of Poisson structures. Classification problems of quantum groups have their analogue in the classification of multiplicative Poisson structures on Lie groups. Complete classifications are difficult and are achieved only in very few cases. For example, given Poisson structures on V and $\wedge^1(V)$, no general rule is known for deciding if the induced multiplicative pre-Poisson structure on $End(V)$ is a Poisson structure (i.e. satisfies the Jacobi-Identity). However, our main result shows that there is such a rule for all following members of the corresponding Clifford sequence (see below). As a consequence we also obtain a new class of triangular multiplicative Poisson structures for the Lie groups $GL(n^2)$.

More exactly, consider the following problem for Poisson vector spaces and Poisson matrix groups.

Start with an n-dimensional Poisson vector space V with coordinates $x_1, \ldots, x_n$ and its dual $\wedge^1(V)$ with anticommuting coordinates $\xi_1, \ldots, \xi_n$. These induce pre-Poisson structures on $End(V)$ and its dual $\wedge^1(End(V))$. Now view the matrix elements of $End(V)$ and $\wedge^1(End(V))$ as constituting the coordinates of a dual pair of n^2-dimensional Poisson vector spaces. These will induce Poisson brackets on $End(End(V))$. Using the terminology of Corrigan *et al* [1] we use for the sequence $V^{(1)} := V$, $V^{(i+1)} := End(V^{(i)})$, $i \geqslant 1$, the notion Clifford sequence of V. The above process will generate Poisson brackets on all $V^{(i)}$.

It is well known that if the Jacobi identities are satisfied on $V^{(1)}$ and $\wedge^1(V^{(1)})$ this will no longer be true for $V^{(2)} = End(V)$. In [2, p L511] Kupershmidt states that no non-trival example is known where the Jacobi idendity is satisfied on both $V^{(2)} = End(V)$ and $V^{(3)} = End(End(V))$.

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We will show here that the Jacobi identity on $V^{(2)}$ will induce the Jacobi identity on $V^{(3)}$ and therefore on all $V^{(i)}$, $i \geqslant 3$ if and only if the Poisson structure is triangular. Therfore even the standard Poisson structure on $GL(n)$ will not induce the Jacobi identity on $V^{(3)}$.

The quantum analogue of this problem for $n = 2$ was suggested by Corrigan *et al* in [1, p 779]. I thank Professor B A Kupershmidt for giving me this problem.

Let us start with the n-dimensional vector space V with coordinates $x_1, \ldots, x_n$ and with the quadratic skewsymmetric bracket

$$\{x_i, x_j\} = c_{ij}^{kl} x_k x_l \qquad c_{ij}^{kl} = -c_{ji}^{kl} = c_{ij}^{lk}. \tag{1}$$

Consider a dual space $\wedge^1(V)$ with anticommuting generators ξ_i and the quadratic symmetric bracket

$$\{\xi_i, \xi_j\} = d_{ij}^{kl} \xi_k \xi_l \qquad d_{ij}^{kl} = d_{ji}^{kl} = -d_{ij}^{lk}. \tag{2}$$

Then there is a unique Poisson structure on $End(V)$:

$$\{M_i^k, M_j^l\} = r_{ij}^{mn} M_m^k M_n^l - r_{mn}^{kl} M_i^m M_j^n \qquad r_{ij}^{kl} = c_{ij}^{kl} + d_{ij}^{kl} \tag{3}$$

such that the actions of $End(V)$ on V and $\wedge^1(V)$ are Poisson actions.

Considering $End(V)$ as an n^2-dimensional vector space we can write

$$\{M_i^k, M_j^l\} = C\begin{smallmatrix} m & n \\ p & q \\ i & j \\ k & l \end{smallmatrix} M_p^m M_q^n \tag{4}$$

where

$$C\begin{smallmatrix} m & n \\ p & q \\ i & j \\ k & l \end{smallmatrix} = \frac{\delta_{kl}^{mn} r_{ij}^{pq} + \delta_{lk}^{mn} r_{ij}^{qp} - \delta_{pq}^{ij} r_{mn}^{kl} - \delta_{pq}^{ji} r_{nm}^{kl}}{2}. \tag{5}$$

Following the idea of Corrigan *et al* the canonical brackets on $\wedge^1(End(V))$ are determined by the condition that the actions

$$\wedge^1(End(V)) \times \wedge^1(V) \to V \qquad \wedge^1\,(End(V)) \times V \to \wedge^1(V) \tag{6}$$

are Poisson. Setting

$$x_i' = \sum_j \Phi_i^j \otimes \xi_j \qquad \xi_i' = \sum_j \Phi_i^j \otimes x_j \tag{7}$$

and requiring that

$$\{x_i', x_j'\}_\otimes = \sum c_{ij}^{kl} x_k' x_l' \qquad \{\xi_i', \xi_j'\}_\otimes = \sum d_{ij}^{kl} \xi_k' \xi_l' \tag{8}$$

($\{\,,\}_\otimes$ denotes the Poisson structure of the direct product on $\wedge^1(End(V)) \times \wedge^1(V)$ and $\wedge^1(End(V)) \times V$, respectively), we obtain

$$\begin{aligned}\{\xi_i', \xi_j'\}_\otimes &= \left\{\sum_k \Phi_i^k \otimes x_k, \sum_l \Phi_j^l \otimes x_l\right\}_\otimes \\ &= \sum_{kl} \{\Phi_i^k, \Phi_j^l\} \otimes x_k x_l + \sum_{kl} \Phi_i^k \Phi_j^l \otimes \sum_{mn} c_{kl}^{mn} x_m x_n \\ &= \sum_{mn} d_{ij}^{mn} \xi_m' \xi_n' = \sum_{mn} d_{ij}^{mn} \sum_{kl} \Phi_m^k \Phi_n^l \otimes x_k x_l \end{aligned} \tag{9}$$

i.e.

$$\{\Phi_i^k, \Phi_j^l\} + \{\Phi_i^l, \Phi_j^k\} = -2 \sum_{mn} c_{mn}^{kl} \Phi_i^m \Phi_j^n + 2 \sum_{mn} d_{ij}^{mn} \Phi_m^k \Phi_n^l. \tag{10}$$

Furthermore

$$\{x'_i, x'_j\}_\otimes = \left\{\sum_k \Phi_i^k \otimes \xi_k, \sum_l \Phi_j^l \otimes \xi_l\right\}_\otimes$$
$$= \sum_{kl}\{\Phi_i^k, \Phi_j^l\} \otimes \xi_k\xi_l + \sum_{kl} \Phi_i^k \Phi_j^l \otimes \sum_{mn} d_{kl}^{mn} \xi_m \xi_n$$
$$= \sum_{mn} c_{ij}^{mn} x'_m x'_n = \sum_{mn} c_{ij}^{mn} \sum_{kl} \Phi_m^k \Phi_n^l \otimes \xi_k \xi_l \tag{11}$$

i.e.

$$\{\Phi_i^k, \Phi_j^l\} - \{\Phi_i^l, \Phi_j^k\} = -2\sum_{mn} d_{mn}^{kl} \Phi_i^m \Phi_j^n + \sum_{mn} c_{ij}^{mn} \left(\Phi_m^k \Phi_n^l + \Phi_m^l \Phi_n^k\right)$$
$$= -2\sum_{mn} d_{mn}^{kl} \Phi_i^m \Phi_j^n + 2\sum_{mn} c_{ij}^{mn} \Phi_m^k \Phi_n^l. \tag{12}$$

Adding equations (10) and (12) we obtain

$$\{\Phi_i^k, \Phi_j^l\} = r_{ij}^{mn} \Phi_m^k \Phi_n^l - r_{mn}^{kl} \Phi_i^m \Phi_j^n \qquad r_{ij}^{kl} = c_{ij}^{kl} + d_{ij}^{kl}. \tag{13}$$

Considering $\wedge^1(End(V))$ as an n^2-dimensional vector space we can write

$$\{\Phi_i^k, \Phi_j^l\} = D\,{}^{m\ n}_{p\ q}{}_{i\ j}{}_{k\ l}\, \Phi_p^m \Phi_q^n. \tag{14}$$

where

$$D\,{}^{m\ n}_{p\ q}{}_{i\ j}{}_{k\ l} = \frac{\delta_{kl}^{mn} r_{ij}^{pq} - \delta_{lk}^{mn} r_{ij}^{qp} - \delta_{pq}^{ij} r_{mn}^{kl} + \delta_{pq}^{ji} r_{nm}^{kl}}{2}. \tag{15}$$

Remark. There is another canonical way to define a dual Poisson bracket. Consider the Poisson space $V, \{.,.\}$. Then the dual structure in the sense of [3] is given by

$$\{\xi_i, \xi_j\} = -c_{kl}^{ij} \xi_k \xi_l. \tag{16}$$

According to (3) equations (5) and (15) result in the following r-matrix for the matrix elements of $End(End(V))$:

$$R\,{}^{m\ n}_{p\ q}{}_{i\ j}{}_{k\ l} = \delta_{kl}^{mn} r_{ij}^{pq} - \delta_{pq}^{ij} r_{mn}^{kl}. \tag{17}$$

The Jacobi identity for $End(V)$ is equivalent to the modified classical Yang–Baxter equation (MCYBE), cf [4]. Let $\{e_i\}$ be a base of V, $\{e^i\}$ be the dual base of V' and $\{e_j^i = e_j \otimes e^i\}$ be a base of $End(V) = V \otimes V'$. We define

$$C(r) = \sum_{ijklmn} C(r)_{ijk}^{lmn} e_l^i \otimes e_m^j \otimes e_n^k$$
$$:= \sum_{ijklmn} \left(r_{jk}^{sn} r_{is}^{lm} - r_{ij}^{ls} r_{sk}^{mn} + r_{ki}^{sl} r_{js}^{mn} - r_{jk}^{ms} r_{si}^{nl} + r_{ij}^{sm} r_{ks}^{nl} - r_{ki}^{ns} r_{sj}^{lm}\right) e_l^i \otimes e_m^j \otimes e_n^k$$

or in short notation

$$C(r)_{ijk}^{lmn} = \left(r_{jk}^{sn} r_{is}^{lm} - r_{ij}^{ls} r_{sk}^{mn} + \mathrm{CP}\right). \tag{18}$$

We have

$$C(r)_{ijk}^{mnp} = C(r)_{jki}^{npm} = C(r)_{kij}^{pmn}. \tag{19}$$

The MCYBE is given by

$$ad_x C(r) = 0 \qquad \forall\, x \in End(V). \tag{20}$$

If we have even $C(r) = 0$, we say r is *triangular* (cf [4]).

On $V^{(3)}$ we have

$$C(R)^{\substack{d\,e\,f\\ l\,m\,n}}_{\substack{i\,j\,k\\ a\,b\,c}} = R^{\substack{t\,f\\ s\,n}}_{\substack{j\,k\\ b\,c}} R^{\substack{d\,e\\ l\,m}}_{\substack{i\,s\\ a\,t}} - R^{\substack{d\,t\\ l\,s}}_{\substack{i\,j\\ a\,b}} R^{\substack{e\,f\\ m\,n}}_{\substack{s\,k\\ t\,c}} + \text{CP}. \tag{21}$$

Inserting (17), the right-hand side of (21) becomes

$$\delta^{def}_{abc}\left(r^{sn}_{jk} r^{lm}_{is} - r^{ls}_{ij} r^{mn}_{sk} + \text{CP}\right) + \delta^{ijk}_{lmn}\left(r^{bc}_{tf} r^{at}_{de} - r^{ab}_{dt} r^{tc}_{ef} + \text{CP}\right)$$

$$+\left(-\delta^{nd}_{ka} r^{bc}_{ef} r^{lm}_{ij} - \delta^{fl}_{ci} r^{mn}_{jk} r^{ab}_{de} + \delta^{fl}_{ci} r^{ab}_{de} r^{mn}_{jk} + \delta^{dn}_{ak} r^{lm}_{ij} r^{bc}_{ef} + \text{CP}\right) \tag{22}$$

the last four terms $+\,\text{CP}$ cancel, i.e.

$$C(R)^{\substack{d\,e\,f\\ l\,m\,n}}_{\substack{i\,j\,k\\ a\,b\,c}} = \delta^{def}_{abc} C(r)^{lmn}_{ijk} - \delta^{ijk}_{lmn} C(r)^{abc}_{def}. \tag{23}$$

Theorem. The MCYBE for R on $V^{(3)}$ is satisfied if and only if r is triangular.

Proof. (i) $C(r) = 0$ is sufficient because of (20) and (23). (ii) Now we show that $C(r) = 0$ is necessary. The MCYBE on $V^{(3)}$ reduces to the vanishing of

$$ad\left(e^{\substack{A\\ I}}_{\substack{L\\ D}}\right) \sum_{\substack{abcdef\\ ijklmn}} C(R)^{\substack{d\,e\,f\\ l\,m\,n}}_{\substack{i\,j\,k\\ a\,b\,c}} e^{\substack{a\\ i}}_{\substack{l\\ d}} \otimes e^{\substack{b\\ j}}_{\substack{m\\ e}} \otimes e^{\substack{c\\ k}}_{\substack{n\\ f}}$$

$$= ad\left(e^{\substack{A\\ I}}_{\substack{L\\ D}}\right) \sum_{\substack{abcdef\\ ijklmn}} \left(\delta^{def}_{abc} C(r)^{lmn}_{ijk} - \delta^{ijk}_{lmn} C(r)^{abc}_{def}\right) e^{\substack{a\\ i}}_{\substack{l\\ d}} \otimes e^{\substack{b\\ j}}_{\substack{m\\ e}} \otimes e^{\substack{c\\ k}}_{\substack{n\\ f}} \tag{24}$$

where

$$e^{\substack{i\\ j}}_{\substack{k\\ l}} := e_k \otimes e^l \otimes e^j \otimes e_i \in End(End(V)) \cong End(V) \otimes End(V)' \cong V \otimes V' \otimes V' \otimes V\,.$$

With

$$ad\left(e^{\substack{A\\ I}}_{\substack{L\\ D}}\right) e^{\substack{a\\ i}}_{\substack{l\\ d}} = \delta^{IA}_{ld} e^{\substack{a\\ i}}_{\substack{L\\ D}} - \delta^{ia}_{LD} e^{\substack{A\\ I}}_{\substack{l\\ d}} \tag{25}$$

we obtain the equation

$$\sum_{\substack{abcef\\ ijkmn}} \left(\delta^{Aef}_{abc} C(r)^{Imn}_{ijk} - \delta^{ijk}_{Imn} C(r)^{abc}_{Aef}\right) e^{\substack{a\\ i}}_{\substack{L\\ D}} \otimes e^{\substack{b\\ j}}_{\substack{m\\ e}} \otimes e^{\substack{c\\ k}}_{\substack{n\\ f}}$$

$$- \sum_{\substack{bcdef\\ jklmn}} \left(\delta^{def}_{Dbc} C(r)^{lmn}_{Ljk} - \delta^{Ljk}_{lmn} C(r)^{Dbc}_{def}\right) e^{\substack{A\\ I}}_{\substack{l\\ d}} \otimes e^{\substack{b\\ j}}_{\substack{m\\ e}} \otimes e^{\substack{c\\ k}}_{\substack{n\\ f}}$$

$$+ \sum_{\substack{abcdf\\ ijkln}} \left(\delta^{dAf}_{abc} C(r)^{lIn}_{ijk} - \delta^{ijk}_{lIn} C(r)^{abc}_{dAf}\right) e^{\substack{a\\ i}}_{\substack{l\\ d}} \otimes e^{\substack{b\\ j}}_{\substack{L\\ D}} \otimes e^{\substack{c\\ k}}_{\substack{n\\ f}}$$

$$- \sum_{\substack{acdef\\ iklmn}} \left(\delta^{def}_{aDc} C(r)^{lmn}_{iLk} - \delta^{iLk}_{lmn} C(r)^{aDc}_{def}\right) e^{\substack{a\\ i}}_{\substack{l\\ d}} \otimes e^{\substack{A\\ I}}_{\substack{m\\ e}} \otimes e^{\substack{c\\ k}}_{\substack{n\\ f}}$$

$$+\sum_{\substack{abcde\\ ijklm}}\left(\delta^{deA}_{abc}C(r)^{lmI}_{ijk}-\delta^{ijk}_{lmI}C(r)^{abc}_{deA}\right)\, e_{\substack{a\\i\\l\\d}}\otimes e_{\substack{b\\j\\m\\e}}\otimes e_{\substack{c\\k\\L\\D}}$$

$$-\sum_{\substack{abdef\\ ijlmn}}\left(\delta^{def}_{abD}C(r)^{lmn}_{ijL}-\delta^{ijL}_{lmn}C(r)^{abD}_{def}\right)\, e_{\substack{a\\i\\l\\d}}\otimes e_{\substack{b\\j\\m\\e}}\otimes e_{\substack{A\\I\\n\\f}}=0. \tag{26}$$

Now consider the coefficient at the base vector

$$e_{\substack{A\\i\\L\\D}}\otimes e_{\substack{b\\j\\m\\b}}\otimes e_{\substack{c\\k\\n\\c}}.$$

For $i\neq I$; $b,c\neq A$; $b,c\neq D$ this coeffient is equal to $C(r)^{Imn}_{ijk}$, i.e.

$$C(r)^{lmn}_{ijk}=0 \qquad \text{if } \ i\neq l. \tag{27}$$

Because of (19) it follows that

$$C(r)^{lmn}_{ijk}=0 \qquad \text{for } \ (i,j,k)\neq(l,m,n). \tag{28}$$

From equation (28) it follows that for the MCYBE

$$\sum_{bcjk}\left(C(r)^{Ijk}_{Ijk}-C(r)^{Ljk}_{Ljk}-C(r)^{Abc}_{Abc}+C(r)^{Dbc}_{Dbc}\right)\, e_{\substack{A\\I\\L\\D}}\otimes e_{\substack{b\\j\\j\\b}}\otimes e_{\substack{c\\k\\k\\c}}$$

$$+\sum_{acik}\left(C(r)^{iIk}_{iIk}-C(r)^{iLk}_{iLk}-C(r)^{aAc}_{aAc}+C(r)^{aDc}_{aDc}\right)\, e_{\substack{a\\i\\i\\a}}\otimes e_{\substack{A\\I\\L\\D}}\otimes e_{\substack{c\\k\\k\\c}}$$

$$+\sum_{abij}\left(C(r)^{ijI}_{ijI}-C(r)^{ijL}_{ijL}-C(r)^{abA}_{abA}+C(r)^{abD}_{abD}\right)\, e_{\substack{a\\i\\i\\a}}\otimes e_{\substack{b\\j\\j\\b}}\otimes e_{\substack{A\\I\\L\\D}}=0. \tag{29}$$

Now consider the coefficient at the base vector

$$e_{\substack{A\\I\\L\\A}}\otimes e_{\substack{b\\j\\j\\b}}\otimes e_{\substack{c\\k\\k\\c}}.$$

For $b,c\neq A$ this coefficient is equal to $C(r)^{Ijk}_{Ijk}-C(r)^{Ljk}_{Ljk}$. Therefore, because of (19), all coefficients $C(r)^{ijk}_{ijk}$ are equal. On the other hand, from equation (18) it follows that $C(r)^{iii}_{iii}=0$. We conclude that

$$C(r)^{ijk}_{ijk}=0 \qquad \forall\, i,j,k. \tag{30}$$

Equations (28) and (30) imply $C(r)=0$. □

Example. (i) In the case of the standard multiplictive Poisson structure

$$r^{kl}_{ij}=\delta^{lk}_{ij}\,\mathrm{sgn}(i-j) \tag{31}$$

we obtain

$$C(r)^{mnp}_{ijk}=\delta^{npm}_{ijk}\big(\mathrm{sgn}(j-k)\,\mathrm{sgn}(i-k)+\mathrm{sgn}(k-i)\,\mathrm{sgn}(j-i)$$

$$+\mathrm{sgn}(i-j)\,\mathrm{sgn}(k-j)\big)-\delta^{pmn}_{ijk}\big(\mathrm{sgn}(i-j)\,\mathrm{sgn}(i-k)$$

$$+\mathrm{sgn}(j-k)\,\mathrm{sgn}(j-i)+\mathrm{sgn}(k-i)\,\mathrm{sgn}(k-j)\big) \tag{32}$$

i.e.

$$C(r)^{211}_{112}=1\neq 0 \tag{33}$$

and therefore we have no Jacobi identity for the induced structure on $V^{(3)}$.
(ii) The r-matrices

$$r_{ij}^{kl} = \delta_{ij}^{kl} \Phi_{ij} \tag{34}$$

are triangular and induce triangular Poisson structures on all $V^{(i)}$, $i \geqslant 3$.

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